



Albert Einstein Institute, Hannover

The First Observation of a Binary Black Hole Merger:  
Status and Future Prospects

## PROBLEM OF MOTION IN GR & POST-NEWTONIAN THEORY

Luc Blanchet

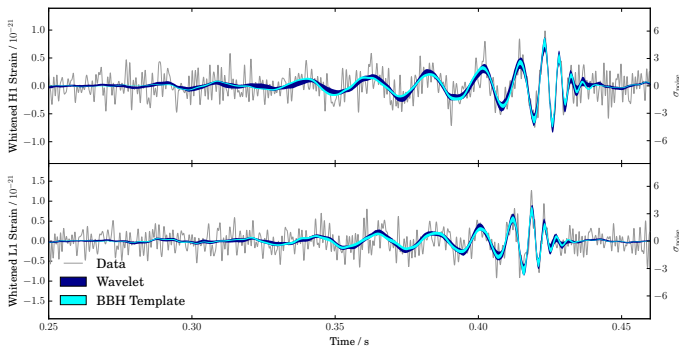
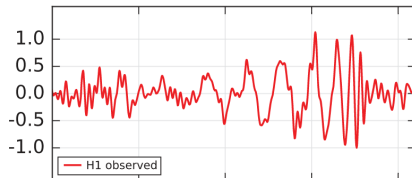
Gravitation et Cosmologie (GR<sub>ε</sub>CO)  
Institut d'Astrophysique de Paris

24 mai 2016

# Binary black-hole event GW150914

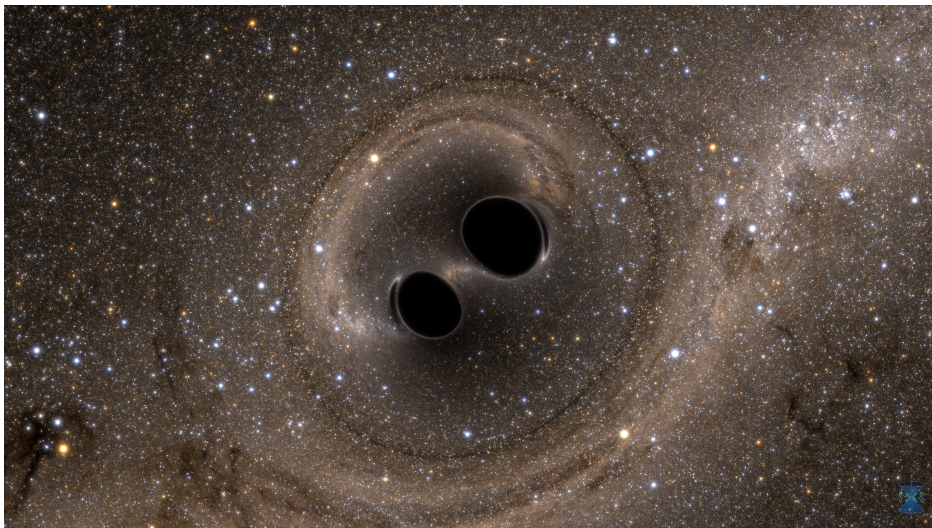
[LIGO/VIRGO collaboration 2016; see talk by Duncan Brown]

Hanford, Washington (H1)



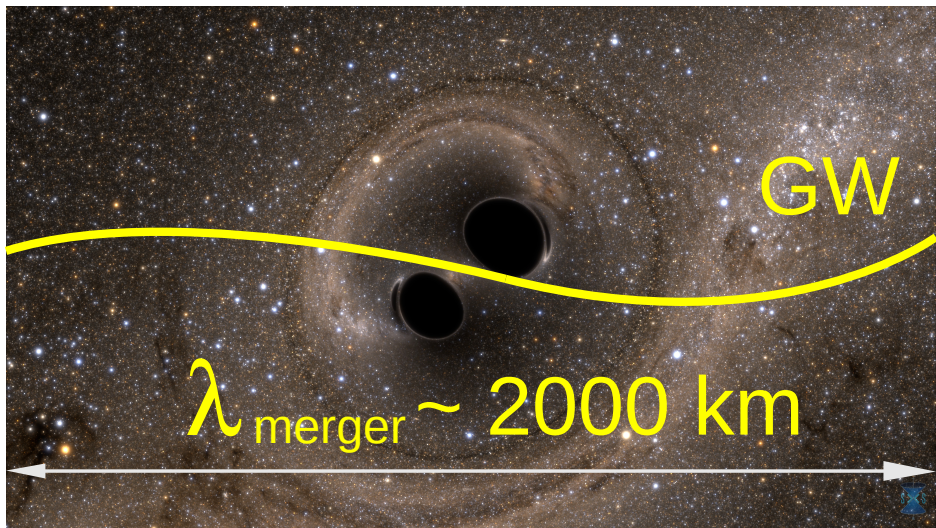
# Binary black-hole event GW150914

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# 100 years of gravitational radiation [Einstein 1916]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

## Näherungsweise Integration der Feldgleichungen der Gravitation.

VON A. EINSTEIN.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die  $g_{\mu\nu}$  in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable  $x_4 = it$  aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$

definierten Größen  $\gamma_{\mu\nu}$ , welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist  $\delta_{\mu\mu} = 1$  bzw.  $\delta_{\mu\nu} = 0$ , je nachdem  $\mu = \nu$  oder  $\mu \neq \nu$ .

Wir werden zeigen, daß diese  $\gamma_{\mu\nu}$  in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.



⇐ small perturbation of Minkowski's metric

# 100 years of gravitational radiation [Einstein 1918]

## Einstein's quadrupole formula

mit  $4\pi R^2$  multiplizierte  $S$  endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

[31]

$$4\pi R^2 \bar{S} = \frac{r}{80\pi} \left[ \sum_{i,j} \ddot{Q}_{ij}^2 - \frac{1}{3} \left( \sum_{i,j} \ddot{Q}_{ij} \right)^2 \right]. \quad (30)$$

[32]

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

[33]

Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont worden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gravitationstheorie wird bringen müssen.

### § 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der



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factor 1/80 should be 1/40

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# Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

## 1 First quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 D} \left\{ \frac{d^2 Q_{ij}}{dt^2} \left( t - \frac{D}{c} \right) + \mathcal{O} \left( \frac{v}{c} \right) \right\}^{\text{TT}} + \mathcal{O} \left( \frac{1}{D^2} \right)$$

## 2 Einstein quadrupole formula

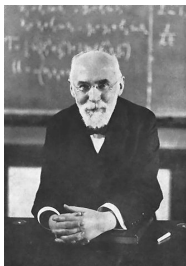
$$\left( \frac{dE}{dt} \right)^{\text{GW}} = \frac{G}{5c^5} \left\{ \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} + \mathcal{O} \left( \frac{v}{c} \right)^2 \right\}$$

## 3 Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5} \rho x^j \frac{d^5 Q_{ij}}{dt^5} + \mathcal{O} \left( \frac{v}{c} \right)^7$$

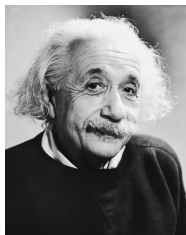
which is a  $2.5\text{PN} \sim (v/c)^5$  effect in the source's equations of motion

# The 1PN equations of motion [Lorentz & Droste 1917]



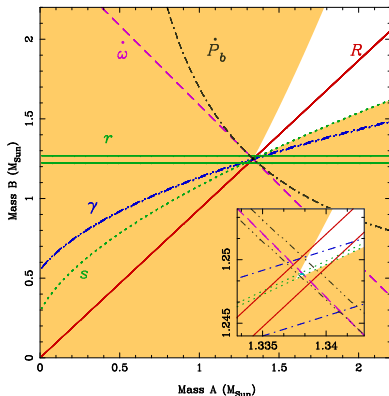
- Obtain the equations of motion of  $N$  bodies at the 1PN  $\sim (v/c)^2$  order and even derive the 1PN Lagrangian!
- This work published in Dutch has been largely unrecognized until an English translation was published in 1937

# The 1PN equations of motion [Einstein, Infeld & Hoffmann 1938]



$$\frac{d^2 \mathbf{r}_A}{dt^2} = - \sum_{B \neq A} \frac{G m_B}{r_{AB}^2} \mathbf{n}_{AB} \left[ 1 - 4 \sum_{C \neq A} \frac{G m_C}{c^2 r_{AC}} - \sum_{D \neq B} \frac{G m_D}{c^2 r_{BD}} \left( 1 - \frac{\mathbf{r}_{AB} \cdot \mathbf{r}_{BD}}{r_{BD}^2} \right) + \frac{1}{c^2} \left( \mathbf{v}_A^2 + 2 \mathbf{v}_B^2 - 4 \mathbf{v}_A \cdot \mathbf{v}_B - \frac{3}{2} (\mathbf{v}_B \cdot \mathbf{n}_{AB})^2 \right) \right] + \sum_{B \neq A} \frac{G m_B}{c^2 r_{AB}^2} \mathbf{v}_{AB} [\mathbf{n}_{AB} \cdot (3 \mathbf{v}_B - 4 \mathbf{v}_A)] - \frac{7}{2} \sum_{B \neq A} \sum_{D \neq B} \frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3} \mathbf{n}_{BD}$$

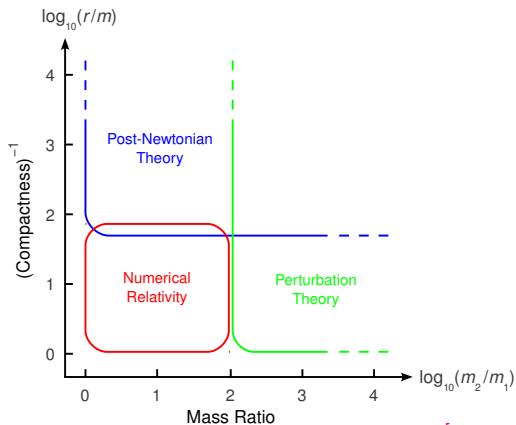
# Relativistic effects in binary pulsars [e.g. Stairs 2003]



- 1PN order {  $\dot{\omega}$  relativistic advance of periastron  
 $\gamma$  gravitational red-shift and second-order Doppler effect  
 $r$  and  $s$  range and shape of the Shapiro time delay
- 2.5PN order {  $\dot{P}$  secular decrease of orbital period

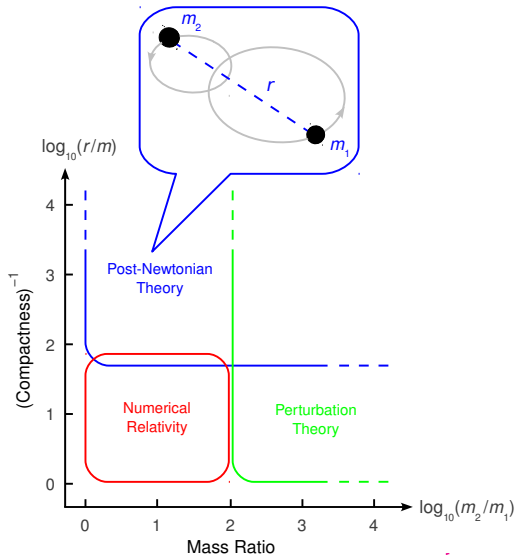


# Methods to compute GW templates



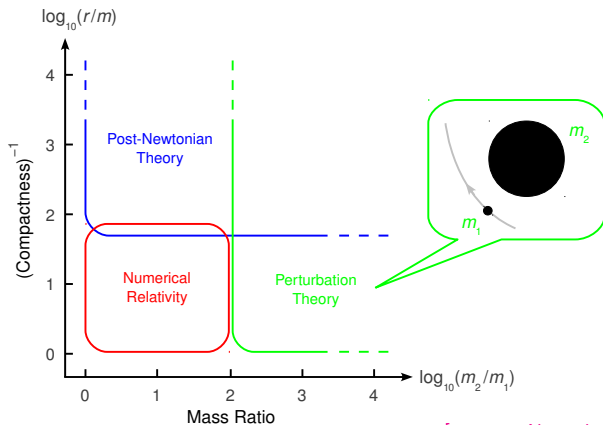
[courtesy Alexandre Le Tiec]

# Methods to compute GW templates



[courtesy Alexandre Le Tiec]

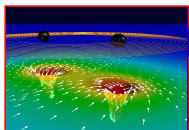
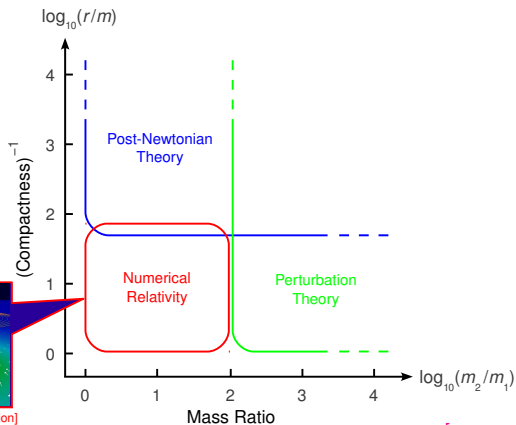
# Methods to compute GW templates



[courtesy Alexandre Le Tiec]

# Methods to compute GW templates

[see talk by Mark Hannam]

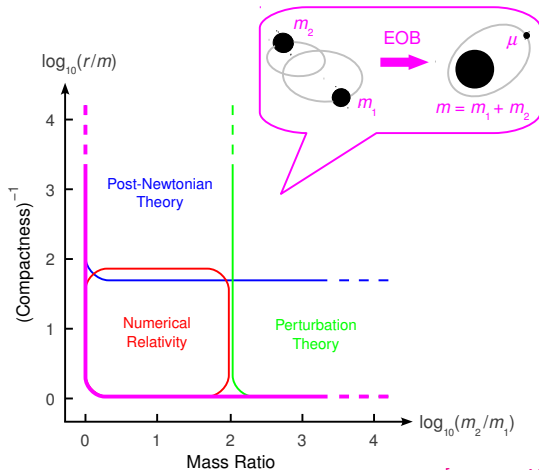


[Caltech/Cornell/CITA collaboration]

[courtesy Alexandre Le Tiec]

# Methods to compute GW templates

[Buonanno & Damour 1998; see talk by Alessandro Nagar]



[courtesy Alexandre Le Tiec]

# Inspiralling binaries require high-order PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]

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PHYSICAL REVIEW LETTERS

17 MAY 1993

## The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

Curt Cutler,<sup>(1)</sup> Theodoris A. Apostolatos,<sup>(1)</sup> Lars Bildsten,<sup>(1)</sup> Lee Samuel Finn,<sup>(2)</sup> Eanna E. Flanagan,<sup>(1)</sup> Daniel Kennefick,<sup>(1)</sup> Dragoljub M. Markovic,<sup>(1)</sup> Amos Ori,<sup>(1)</sup> Eric Poisson,<sup>(1)</sup> Gerald Jay Sussman,<sup>(1),(a)</sup> and Kip S. Thorne<sup>(1)</sup>

<sup>(1)</sup>Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125  
<sup>(2)</sup>Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208

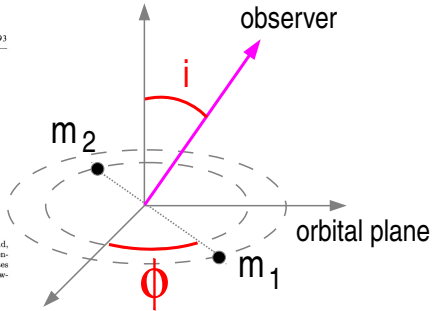
(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy  $\ll 10^{-3}$  and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network

as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does not mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \underbrace{\phi_0 - \frac{M}{\mu} \left( \frac{GM\omega}{c^3} \right)^{-5/3}}_{\text{quadrupole formalism}} \left\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \dots + \frac{3\text{PN}}{c^6} + \dots}_{\text{needs to be computed with 3PN precision at least}} \right\}$$

Here 3PN means **5.5PN** as a radiation reaction effect !

# The intermediate binary black hole problem

PHYSICAL REVIEW D, VOLUME 58, 061501

## Computing the merger of black-hole binaries: The IBBH problem

Patrick R. Brady, Jolien D. E. Creighton, and Kip S. Thorne  
*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125*  
(Received 22 April 1998; published 26 August 1998)

Gravitational radiation arising from the inspiral and merger of binary black holes (BBH's) is a promising candidate for detection by kilometer-scale interferometric gravitational wave observatories. This Rapid Communication discusses a serious obstacle to searches for such radiation and to the interpretation of any observed waves: the inability of current computational techniques to evolve a BBH through its last  $\sim 10$  orbits of inspiral ( $\sim 100$  radians of gravitational-wave phase). A new set of numerical-relativity techniques is proposed for solving this "intermediate binary black hole" (IBBH) problem: (i) numerical evolutions performed in coordinates co-rotating with the BBH, in which the metric coefficients evolve on the long timescale of inspiral, and (ii) techniques for mathematically freezing out gravitational degrees of freedom that are not excited by the waves. [S0556-2821(98)50218-4]

PACS number(s): 04.25.Dm, 04.30.Db, 04.70.-s

### I. MOTIVATION

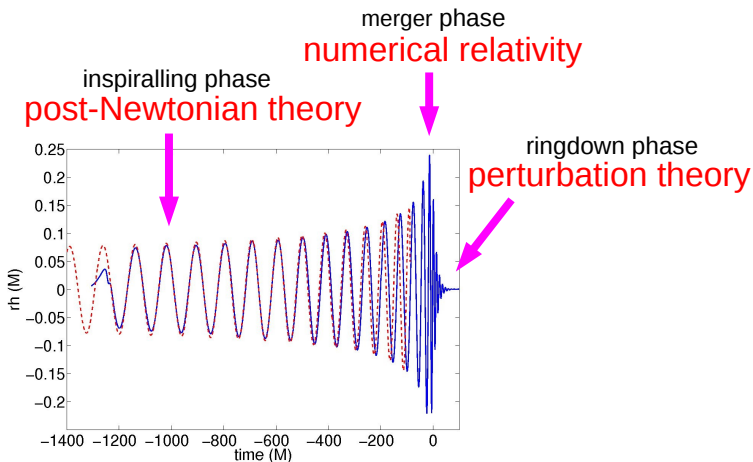
Among all gravitational wave sources that theorists have considered, the one most likely to be detected first is the final inspiral and merger of binary black holes (BBH's) with

that, in the next several years, this approach will be able to evolve a BBH through the gap for the required  $\geq 1200$  dynamical time scales. This motivates exploring alternative procedures for computing the evolution and waves during the IBBH phase.

- An alternative solution is to extend the region of validity of the PN approximation by using Padé approximants [Damour, Iyer & Sathyaprakash 1998]
- However the accuracy of the PN approximation for comparable masses turned out to be rather good far into the strong field region [Blanchet 2001]

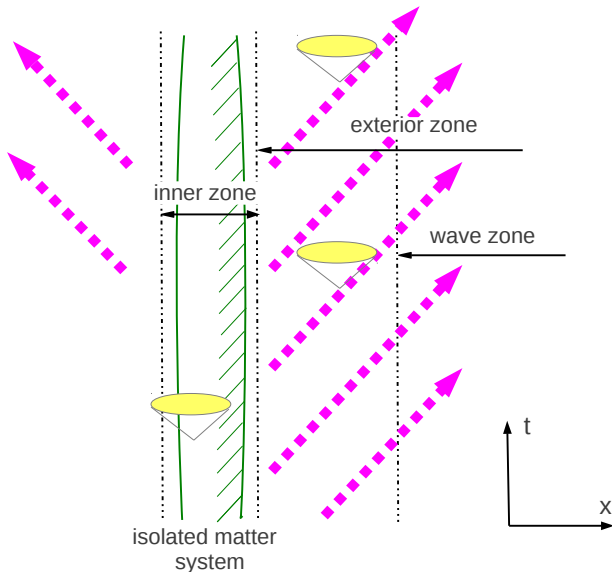


# The gravitational chirp of compact binaries

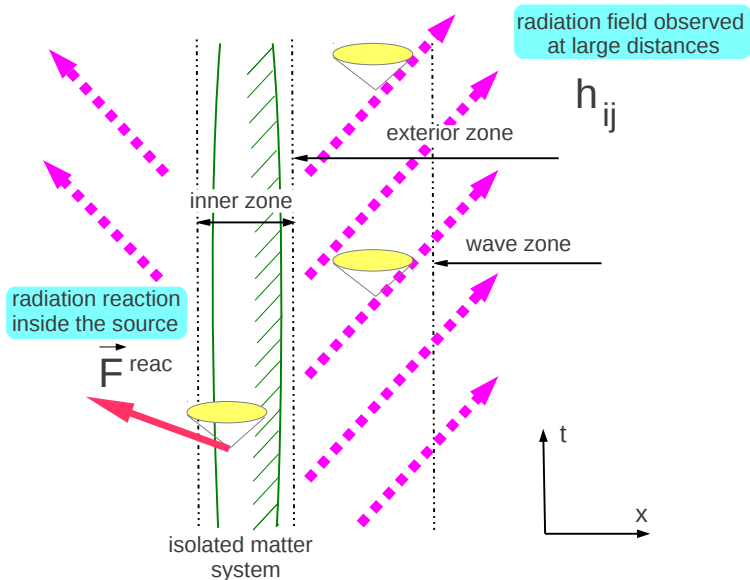


Effective methods such as EOB that interpolate between the PN and NR are very important notably for the data analysis

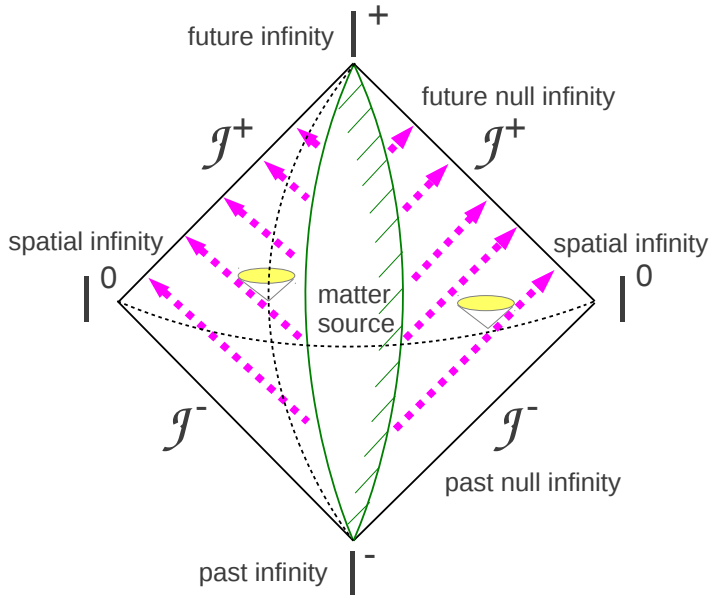
# Isolated matter system in general relativity



# Isolated matter system in general relativity



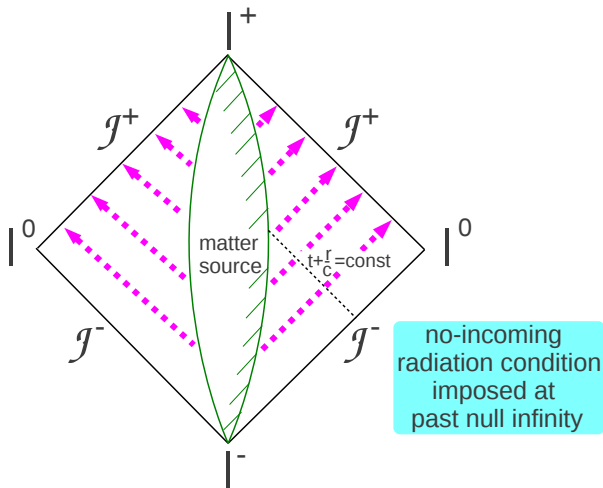
# Conformal picture [Penrose 1963]



# Asymptotic structure of space-time

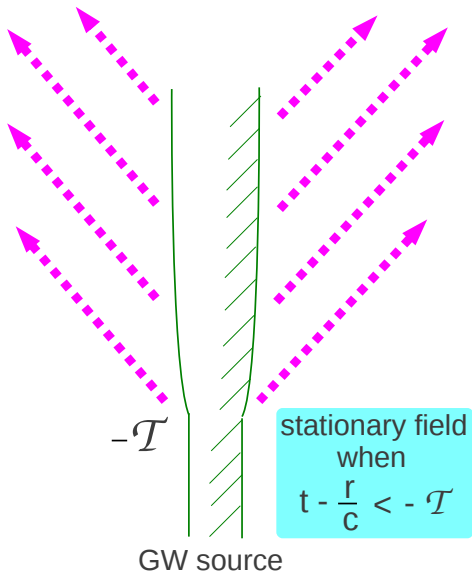
- 1 What is the structure of space-time far away from an isolated matter system?
- 2 Does a general radiating space-time satisfy rigorous definitions [Penrose 1963, 1965] of asymptotic flatness in general relativity?
- 3 How to relate the asymptotic structure of space-time [Bondi *et al.* 1962; Sachs 1962] to the matter variable and dynamics of an actual source?
- 4 How to impose rigorous boundary conditions on the edge of space-time appropriate to an isolated system?

# No-incoming radiation condition



$$\lim_{\substack{r \rightarrow +\infty \\ t + \frac{r}{c} = \text{const}}} \left( \frac{\partial}{\partial r} + \frac{\partial}{c \partial t} \right) (r h^{\alpha\beta}) = 0$$

# Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (e.g. a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past  $-\mathcal{T}$



# Post-Minkowskian expansion

Appropriate for **weakly self-gravitating** isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \quad \left\{ \begin{array}{l} M \text{ mass of source} \\ a \text{ size of source} \end{array} \right.$$

$$\mathbf{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n h_{(n)}^{\alpha\beta}}_{G \text{ labels the PM expansion}}$$

$$\begin{aligned} \square h_{(n)}^{\alpha\beta} &= \frac{16\pi G}{c^4} |g| T_{(n)}^{\alpha\beta} + \overbrace{\Lambda_{(n)}^{\alpha\beta} [h_{(1)}, \dots, h_{(n-1)}]}^{\text{know from previous iterations}} \\ \partial_\mu h_{(n)}^{\alpha\mu} &= 0 \end{aligned}$$

Very difficult approximation to implement in practice for general sources at high PM orders [e.g. Thorne & Kovács 1975]

## Linearized multipolar vacuum solution [Thorne 1980]

General solution of linearized vacuum field equations in harmonic coordinates

$$\square h_{(1)}^{\alpha\beta} = \partial_\mu h_{(1)}^{\alpha\mu} = 0$$

$$h_{(1)}^{00} = -\frac{4}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left( \frac{1}{r} M_L(u) \right)$$

$$h_{(1)}^{0i} = \frac{4}{c^3} \sum_{\ell=1}^{+\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-1} \left( \frac{1}{r} M_{iL-1}^{(1)}(u) \right) + \frac{\ell}{\ell+1} \epsilon_{iab} \partial_{aL-1} \left( \frac{1}{r} S_{bL-1}(u) \right) \right\}$$

$$h_{(1)}^{ij} = -\frac{4}{c^4} \sum_{\ell=2}^{+\infty} \frac{(-)^\ell}{\ell!} \left\{ \partial_{L-2} \left( \frac{1}{r} M_{ijL-2}^{(2)}(u) \right) + \frac{2\ell}{\ell+1} \partial_{aL-2} \left( \frac{1}{r} \epsilon_{ab(i} S_{j)bL-2}^{(1)}(u) \right) \right\}$$

- multipole moments  $M_L(u)$  and  $S_L(u)$  arbitrary functions of  $u = t - r/c$
- mass  $M = \text{const}$ , center-of-mass position  $X_i \equiv M_i/M = \text{const}$ , linear momentum  $P_i \equiv M_i^{(1)} = 0$ , angular momentum  $S_i = \text{const}$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

- 1 The linearized solution is the starting point of an **explicit MPM algorithm**

$$h_{\text{MPM}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_{(n)}^{\alpha\beta}$$

- 2 Hierarchy of perturbation equations is solved by induction over  $n$

$$\square h_{(n)}^{\alpha\beta} = \Lambda_{(n)}^{\alpha\beta}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}]$$
$$\partial_\mu h_{(n)}^{\alpha\mu} = 0$$

- 3 A **regularization** is required in order to cope with the divergency of the multipolar expansion when  $r \rightarrow 0$

# Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

## Theorem 1:

The MPM solution is the **most general solution** of Einstein's vacuum equations outside an isolated matter system

## Theorem 2:

The general structure of the PN expansion is

$$h_{\text{PN}}^{\alpha\beta}(\mathbf{x}, t, c) = \sum_{\substack{p \geq 2 \\ q \geq 0}} \frac{(\ln c)^q}{c^p} h_{p,q}^{\alpha\beta}(\mathbf{x}, t)$$

## Theorem 3:

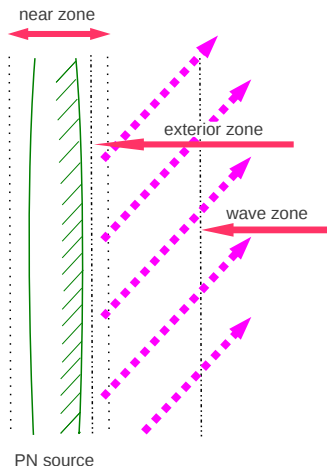
The MPM solution is **asymptotically simple at future null infinity** in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

$$\underbrace{M_{\text{B}}(u)}_{\text{Bondi mass}} = \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5} \int_{-\infty}^u d\tau M_{ij}^{(3)}(\tau) M_{ij}^{(3)}(\tau)$$

+ higher multipoles and higher PM computable to any order

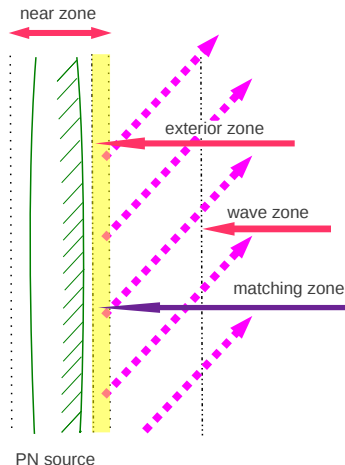
# The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



# The MPM-PN formalism

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



# The matching equation

- 1 This is a variant of the **theory of matched asymptotic expansions**  
[Kates 1980; Anderson *et al.* 1982]

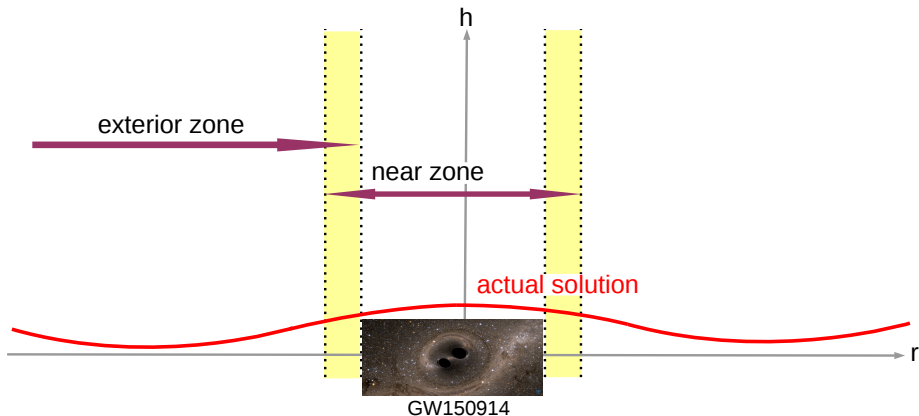
$$\text{match} \quad \left\{ \begin{array}{l} \text{the multipole expansion } \mathcal{M}(h^{\alpha\beta}) \equiv h_{\text{MPM}}^{\alpha\beta} \\ \text{with} \\ \text{the PN expansion } \bar{h}^{\alpha\beta} \equiv h_{\text{PN}}^{\alpha\beta} \end{array} \right.$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

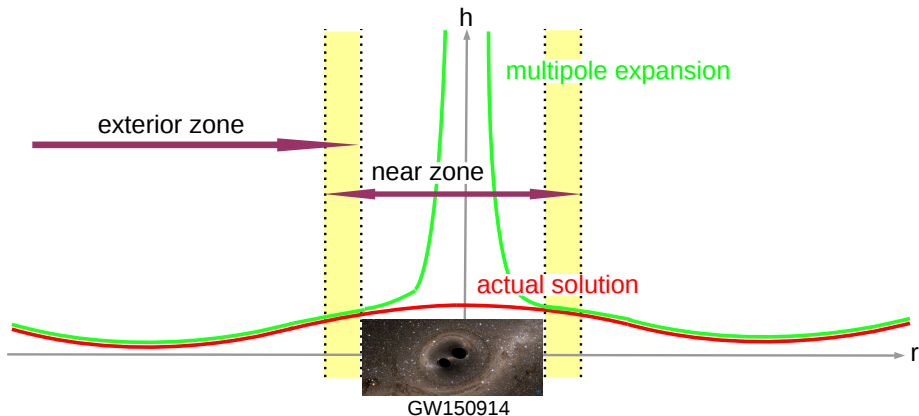
- Left side is the NZ expansion ( $r \rightarrow 0$ ) of the exterior MPM field
- Right side is the FZ expansion ( $r \rightarrow \infty$ ) of the inner PN field
- 2 The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
- 3 It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source



# The matching equation

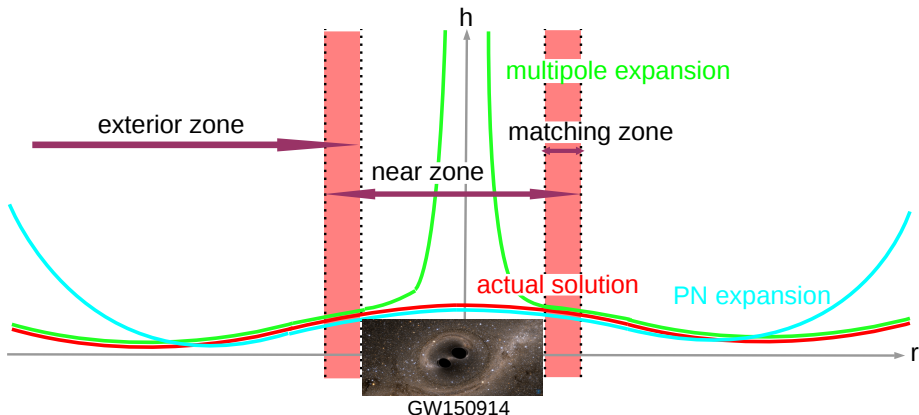


# The matching equation





# The matching equation



# General solution for the multipolar field

$$\mathcal{M}(h^{\mu\nu}) = \text{FP} \square_{\text{ret}}^{-1} \mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{M_L^{\mu\nu}(t - r/c)}{r} \right\}}_{\text{homogeneous retarded solution}}$$

where 
$$M_L^{\mu\nu}(t) = \text{FP} \int d^3\mathbf{x} \hat{x}_L \int_{-1}^1 dz \delta_\ell(z) \underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x}, t - zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The **FP** procedure plays the role of an **UV regularization** in the non-linearity term but an **IR regularization** in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem

## General solution for the inner PN field

$$\bar{h}^{\mu\nu} = \text{FP} \square_{\text{ret}}^{-1} \bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{R_L^{\mu\nu}(t - r/c) - R_L^{\mu\nu}(t + r/c)}{r} \right\}}_{\text{homogeneous antisymmetric solution}}$$

where  $R_L^{\mu\nu}(t) = \text{FP} \int d^3\mathbf{x} \hat{x}_L \int_1^\infty dz \gamma_\ell(z) \underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$

- The **radiation reaction effects** starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects **associated with tails** are contained in the second term and start at 4PN order

# Radiative moments at future null infinity

Correct for the **logarithmic deviation** of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{T - \frac{R}{c}}_{\text{radiative coordinates}} = \underbrace{t - \frac{r}{c}}_{\text{harmonic coordinates}} - \frac{2GM}{c^3} \ln \left( \frac{r}{c\tau_0} \right) + \mathcal{O} \left( \frac{1}{r} \right)$$

Asymptotic waveform is parametrized by **radiative moments**  $U_L$  and  $V_L$  [Thorne 1980]

$$h_{ij}^{\text{TT}} = \frac{1}{R} \sum_{\ell=2}^{\infty} N_{L-2} \underbrace{U_{ijL-2}(T - R/c)}_{\text{mass-type}} + \epsilon_{ab(i} N_{aL-1} \underbrace{V_{j)bL-2}(T - R/c)}_{\text{current-type}} + \mathcal{O} \left( \frac{1}{R^2} \right)$$

# The 3PN radiative quadrupole moment

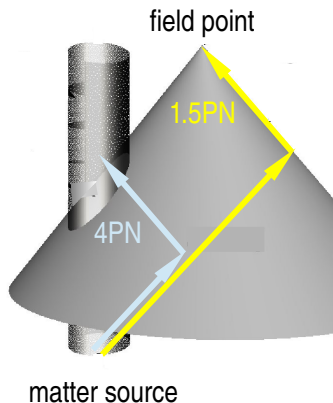
$$\begin{aligned}
 U_{ij}(t) = & M_{ij}^{(2)}(t) + \underbrace{\frac{2GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)}(t-\tau) \left[ \ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12} \right]}_{\text{1.5PN tail integral}} \\
 & + \frac{G}{c^5} \left\{ \underbrace{-\frac{2}{7} \int_0^{+\infty} d\tau M_{a<i}^{(3)} M_{j>a}^{(3)}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms} \right\} \\
 & + \underbrace{\frac{2G^2 M^2}{c^6} \int_0^{+\infty} d\tau M_{ij}^{(5)}(t-\tau) \left[ \ln^2\left(\frac{\tau}{2\tau_0}\right) + \frac{57}{70} \ln\left(\frac{\tau}{2\tau_0}\right) + \frac{124627}{44100} \right]}_{\text{3PN tail-of-tail integral}} \\
 & + \mathcal{O}\left(\frac{1}{c^7}\right)
 \end{aligned}$$

Presence of tail-of-tail-of-tail effect at 4.5PN order [Marchand, Blanchet & Faye, in progress]



# Tails of gravitational waves [Bonnor 1959; Blanchet & Damour 1988, 1992]

Tails are produced by backscatter of GWs on the curvature induced by the matter source's total mass  $M$



$$\delta h_{ij}^{\text{tail}} = \frac{4G}{c^4 r} \underbrace{\frac{GM}{c^3} \int_{-\infty}^t dt' M_{ij}(t') \ln\left(\frac{t-t'}{\tau_0}\right)}_{\text{The tail is dominantly a 1.5PN effect}} + \dots$$

# Application to compact binary inspiral

- 1 Apply the previous PN solution to systems of point particles

$$T^{\mu\nu}(x) = \sum_A \int_{-\infty}^{+\infty} d\tau_A p_A^{(\mu} u_A^{\nu)} \frac{\delta^{(4)}(x - y_A)}{\sqrt{-g_A}} + (\text{spin contributions})$$

- 2 Supplement the calculation by a self-field regularization
  - Hadamard's regularization
  - Dimensional regularization
- 3 The self-field regularization should be applied conjointly with the FP regularization, say in the multipole moments

$$\text{FP}_{B \rightarrow 0} \left\{ \text{AC}_{d \rightarrow 3} \int \frac{d^d \mathbf{x}}{\ell_0^{d-3}} \left( \frac{|\mathbf{x}|}{r_0} \right)^B F(\mathbf{x}) \right\}$$

- 4 The IR scale  $r_0$  and UV scale  $\ell_0$  should disappear at the end of the calculation

# Dimensional regularization

- 1 Einstein's field equations are solved in  $d$  spatial dimensions (with  $d \in \mathbb{C}$ ) with distributional sources. In Newtonian approximation

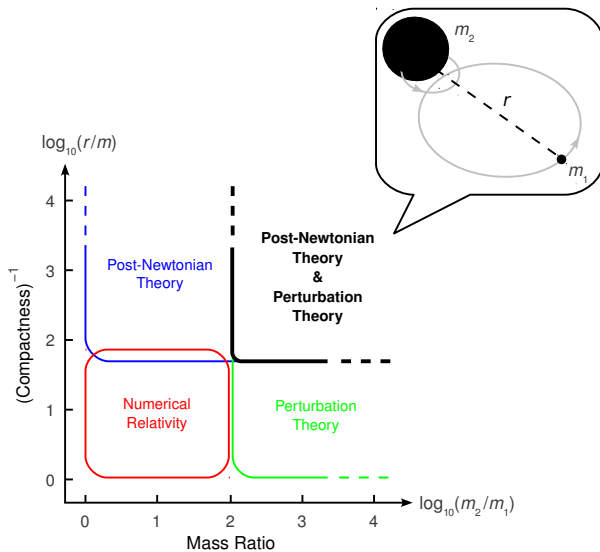
$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

- 2 For two point-particles  $\rho = m_1\delta_{(d)}(\mathbf{x} - \mathbf{y}_1) + m_2\delta_{(d)}(\mathbf{x} - \mathbf{y}_2)$  we get

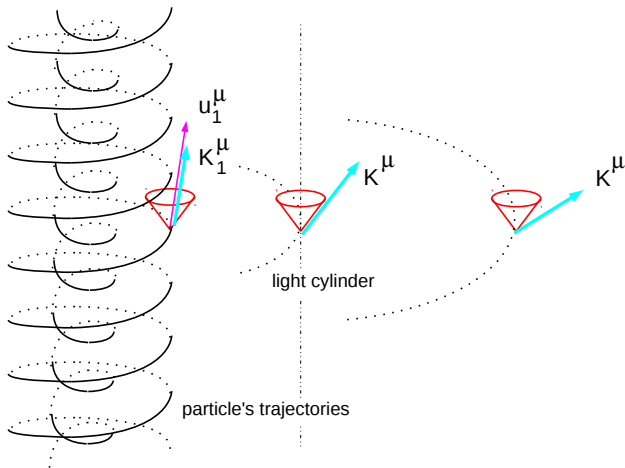
$$U(\mathbf{x}, t) = \frac{2(d-2)k}{d-1} \left( \frac{Gm_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{Gm_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}} \right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}$$

- 3 Computations are performed when  $\Re(d)$  is a large negative number, and the result is **analytically continued** for any  $d \in \mathbb{C}$  except for isolated poles
- 4 Dimensional regularization is then followed by a **renormalization** of the worldline of the particles so as to absorb the poles  $\propto (d-3)^{-1}$

# Checking the PN machinery with GSF



# Looking at the conservative part of the dynamics



$$u_1^\mu = u_1^t K^\mu \quad \text{where} \quad u_1^t = \frac{1}{z_1} = \left( - \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{-1/2} \quad [\text{Detweiler 2008}]$$

## Standard PN theory agrees with GSF calculations

$$\begin{aligned}
 u_{\text{SF}}^t = & -y - 2y^2 - 5y^3 + \left( -\frac{121}{3} + \frac{41}{32}\pi^2 \right) y^4 \\
 & + \left( -\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_E - \frac{64}{5}\ln(16y) \right) y^5 \\
 & - \frac{956}{105}y^6 \ln y - \frac{13696\pi}{525}y^{13/2} - \frac{51256}{567}y^7 \ln y + \frac{81077\pi}{3675}y^{15/2} \\
 & + \frac{27392}{525}y^8 \ln^2 y + \frac{82561159\pi}{467775}y^{17/2} - \frac{27016}{2205}y^9 \ln^2 y \\
 & - \frac{11723776\pi}{55125}y^{19/2} \ln y - \frac{4027582708}{9823275}y^{10} \ln^2 y \\
 & + \frac{99186502\pi}{1157625}y^{21/2} \ln y + \frac{23447552}{165375}y^{11} \ln^3 y + \dots
 \end{aligned}$$

- **Integral PN terms** such as 3PN permit checking dimensional regularization [Blanchet, Detweiler, Le Tiec & Whiting 2010]
- **Half-integral PN terms** starting at 5.5PN order permit checking the non-linear tail (and tail-of-tail) terms [Blanchet, Faye & Whiting 2014]

## 3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$\begin{aligned}
 \mathcal{F} = & \frac{32c^5}{5G} \nu^2 x^5 \left\{ 1 + \overbrace{\left( -\frac{1247}{336} - \frac{35}{12} \nu \right)}^{1\text{PN}} x + \overbrace{4\pi x^{3/2}}^{1.5\text{PN tail}} \right. \\
 & + \left( -\frac{44711}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 + \overbrace{\left( -\frac{8191}{672} - \frac{583}{24} \nu \right) \pi x^{5/2}}^{2.5\text{PN tail}} \\
 & + \left[ \frac{6643739519}{69854400} + \overbrace{\left( \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma_E - \frac{856}{105} \ln(16x) \right)}^{3\text{PN tail-of-tail}} \right. \\
 & \quad \left. + \left( -\frac{134543}{7776} + \frac{41}{48} \pi^2 \right) \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right] x^3 \\
 & + \overbrace{\left( -\frac{16285}{504} + \frac{214745}{1728} \nu + \frac{193385}{3024} \nu^2 \right) \pi x^{7/2}}^{3.5\text{PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \left. \right\}
 \end{aligned}$$

## 3.5PN energy flux of compact binaries

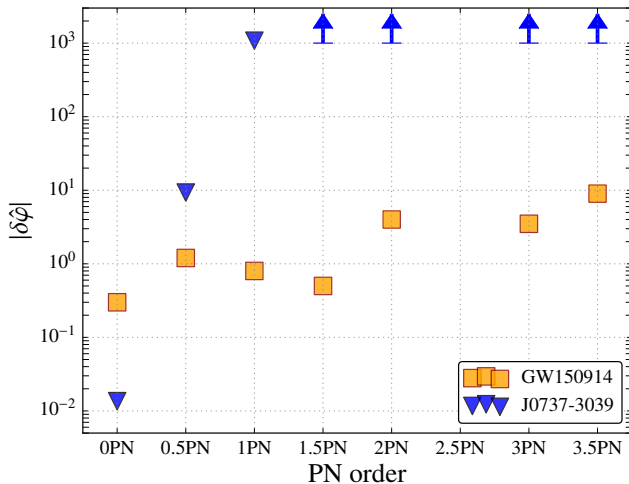
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 \end{aligned}$$



# Bounds on PN parameters with GW150914

[LIGO/VIRGO collaboration 2016; see talk by Chris van den Broeck]



## 3.5PN dominant gravitational wave modes

[BIWW 1995; ABIQ 2004; BFIS 2008; FBI 2014]

$$h_{22} = \frac{2G m \nu x}{R c^2} \sqrt{\frac{16\pi}{5}} e^{-2i\psi} \left\{ 1 + x \left( -\frac{107}{42} + \frac{55\nu}{42} \right) + 2\pi x^{3/2} \right. \\ \left. + x^2 \left( -\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512} \right) \right. \\ \left. + \underbrace{[\dots] x^{5/2}}_{2.5\text{PN}} + \underbrace{[\dots] x^3}_{3\text{PN}} + \underbrace{[\dots] x^{7/2}}_{3.5\text{PN}} + \mathcal{O}(x^4) \right\}$$

$h_{33} = \dots$   
 $h_{31} = \dots$

Tail contributions in this expression are factorized out in the phase variable

$$\psi = \phi - \frac{2GM\omega}{c^3} \ln \left( \frac{\omega}{\omega_0} \right)$$

# 4PN spin-orbit effects in the orbital frequency

[Marsat, Bohé, Faye, Blanchet & Buonanno 2013]

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5} \nu x^{5/2} \left\{ \overbrace{1 + x [\dots] + x^{3/2} [\dots] + x^2 [\dots] + x^{5/2} [\dots] + x^3 [\dots]}^{\text{non-spin terms}} \right. \\
+ \underbrace{[\dots] x^{3/2}}_{1.5\text{PN SO}} + \underbrace{[\dots] x^2}_{2\text{PN SS}} + \underbrace{[\dots] x^{5/2}}_{2.5\text{PN SO}} + \underbrace{[\dots] x^3}_{3\text{PN SO}_{\text{tail}} \& \text{SS}} \\
+ \underbrace{[\dots] x^{7/2}}_{3.5\text{PN SO}} + \underbrace{[\dots] x^4}_{4\text{PN SO}_{\text{tail}} \& \text{SS}} + \mathcal{O}(x^4) \left. \right\}$$

- Leading SO and SS terms due to [Kidder, Will & Wiseman 1993; Kidder 1995]
- Many NL SS terms in EOM computed with the ADM Hamiltonian [Hergt, Steinhoff & Schäfer 2010] and the Effective Field Theory [Porto & Rothstein 2006; Levi 2010]

# THE 4PN EQUATIONS OF MOTION

*Based on a collaboration with*

Laura Bernard, Alejandro Bohé, Guillaume Faye & Sylvain Marsat

# 4PN equations of motion of compact binaries

$$\begin{aligned}
 \frac{dv_1^i}{dt} = & - \frac{Gm_2}{r_{12}^2} n_{12}^i \\
 & \underbrace{\hspace{10em}}_{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} \\
 & + \frac{1}{c^2} \left\{ \left[ \frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \dots \right] n_{12}^i + \dots \right\} \\
 & + \underbrace{\frac{1}{c^4} [\dots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{\substack{\text{2.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6} [\dots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{\substack{\text{3.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8} [\dots]}_{\substack{\text{4PN} \\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)
 \end{aligned}$$

2PN	{	[Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985]	ADM Hamiltonian
		[Damour & Deruelle 1981; Damour 1983]	Harmonic coordinates
		[Kopeikin 1985; Grishchuk & Kopeikin 1986]	Extended fluid balls
		[Blanchet, Faye & Ponsot 1998]	Direct PN iteration
		[Itoh, Futamase & Asada 2001]	Surface integral method

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 & \underbrace{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} \\
 & + \frac{1}{c^2} \left\{ \left[ \frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \dots \right] n_{12}^i + \dots \right\} \\
 & + \underbrace{\frac{1}{c^4} [\dots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{\substack{\text{2.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6} [\dots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{\substack{\text{3.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8} [\dots]}_{\substack{\text{4PN} \\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)
 \end{aligned}$$

3PN	$\left\{ \begin{array}{l} \text{[Jaranowski \& Schäfer 1999; Damour, Jaranowski \& Schäfer 2001]} \\ \text{[Blanchet \& Faye 2000; de Andrade, Blanchet \& Faye 2001]} \\ \text{[Itoh \& Futamase 2003; Itoh 2004]} \\ \text{[Foffa \& Sturani 2011]} \end{array} \right.$	ADM Hamiltonian
		Harmonic equations of motion
		Surface integral method
		Effective field theory
4PN	$\left\{ \begin{array}{l} \text{[Jaranowski \& Schäfer 2013; Damour, Jaranowski \& Schäfer 2014]} \\ \text{[Bernard, Blanchet, Bohé, Faye \& Marsat 2015]} \end{array} \right.$	ADM Hamiltonian
		Fokker Lagrangian

# Fokker action of $N$ particles [Fokker 1929]



- 1 Gauge-fixed action for a system of  $N$  point particles

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \underbrace{\frac{1}{2} g_{\mu\nu} \Gamma^\mu \Gamma^\nu}_{\text{Gauge-fixing term}} \right] - \sum_A m_A c^2 \underbrace{\int dt \sqrt{-(g_{\mu\nu})_A v_A^\mu v_A^\nu / c^2}}_{N \text{ point particles}}$$

- 2 Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

$$g_{\mu\nu}(\mathbf{x}, t) \longrightarrow \bar{g}_{\mu\nu}(\mathbf{x}; \mathbf{y}_B(t), \mathbf{v}_B(t), \dots)$$

- 3 The PN equations of motion of the  $N$  particles (self-gravitating system) are

$$\boxed{\frac{\delta S_F}{\delta \mathbf{y}_A} \equiv \frac{\partial L_F}{\partial \mathbf{y}_A} - \frac{d}{dt} \left( \frac{\partial L_F}{\partial \mathbf{v}_A} \right) + \dots = 0}$$

# Fokker action in the PN approximation

- The Fokker action is split into a PN (**near-zone**) term plus a contribution involving the multipole (**far-zone**) expansion

$$S_{\text{F}}^g = \text{FP}_{B=0} \int d^4x \left(\frac{r}{r_0}\right)^B \bar{\mathcal{L}}_g + \text{FP}_{B=0} \int d^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)$$

- The multipole term gives zero for any “**instantaneous**” term

$$\int d^4x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)|_{\text{inst}} = 0$$

thus only “**hereditary**” terms contribute and they are at least 5.5PN

- Finally we obtain

$$S_{\text{F}}^g = \text{FP}_{B=0} \int d^4x \left(\frac{r}{r_0}\right)^B \bar{\mathcal{L}}_g$$

where the constant  $r_0$  represents an **IR cut-off scale** and plays a crucial role at the 4PN order



# Gravitational wave tail effect at the 4PN order

- At 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action

$$S_F^{\text{tail}} = \frac{G^2 M}{5c^8} \text{Pf}_{s_0} \iint \frac{dt dt'}{|t - t'|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t')$$

- The constant  $s_0$  is *a priori* different from the IR scale  $r_0$  but posing

$$s_0 = r_0 e^{-\alpha}$$

we find that  $r_0$  finally cancels out so the **result is IR finite**

- The remaining constant  $\alpha$  turns out to be an **ambiguity parameter** that we fix by requiring that the energy invariant function for circular orbits agrees with **gravitational self-force** (GSF) calculations at 4PN order

# Conserved energy for circular orbits at 4PN order

- The energy for circular orbits at the 4PN order in the **small mass ratio limit** is known from **self-force calculations** of the redshift variable [Bini & Damour 2013]
- This permits to **fix the ambiguity parameter  $\alpha$**  and to complete the 4PN equations of motion

$$\begin{aligned}
 E^{4\text{PN}} = & -\frac{\mu c^2 x}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) x + \left( -\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24} \right) x^2 \right. \\
 & + \left( -\frac{675}{64} + \left[ \frac{34445}{576} - \frac{205}{96}\pi^2 \right] \nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3 \right) x^3 \\
 & + \left( -\frac{3969}{128} + \left[ -\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_E + \frac{448}{15}\ln(16x) \right] \nu \right. \\
 & \left. \left. + \left[ -\frac{498449}{3456} + \frac{3157}{576}\pi^2 \right] \nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4 \right) x^4 \right\}
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 \end{aligned}$$

# Conserved energy for circular orbits at 4PN order

- We did several computations of the energy function  $E(\Omega)$ . For instance, we can use the associated Hamiltonian formalism [DJS 2014]

$$S_F = \int_{-\infty}^{+\infty} \left[ \sum_A p_A^i v_A^i - H \right] \quad \text{where} \quad H_F^{\text{tail}} = -L_F^{\text{tail}}$$

- With canonical variables  $r, \varphi, p_r, p_\varphi$  appropriate to the reduced problem we solve for circular orbits

$$\begin{aligned} \frac{\delta H}{\delta r} [r^0, p_r^0 = 0, p_\varphi^0] &= 0 \\ \frac{\delta H}{\delta p_\varphi} [r^0, p_r^0 = 0, p_\varphi^0] &= \Omega \end{aligned}$$

- Our end result for the 4PN equations of motion differs by three terms from the result obtained by means of the ADM Hamiltonian [DJS 2014] and more work is needed to understand this difference [BBBFM, in progress]